Putting a classroom spin on invariance in circles

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An old chestnut goes something like this. The surface area of a pond in the form of an annulus is required, but the only measurement possible is the length of the chord across the outer circumference and tangent to the inner circumference as shown in Figure 1.
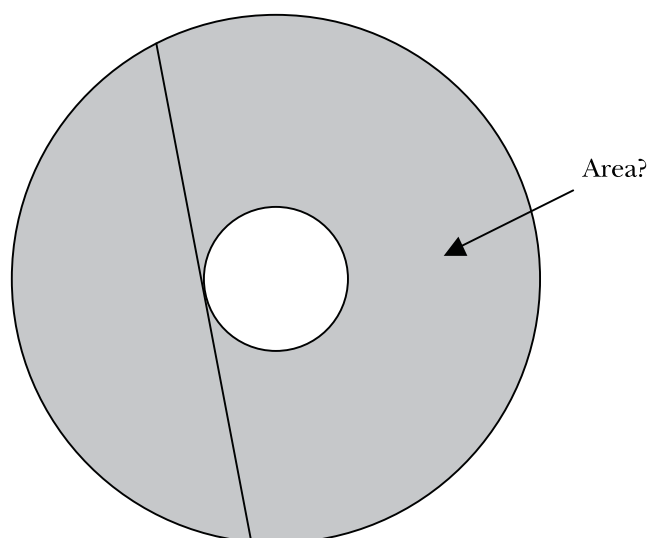

Figure 1. Pond in form of an annulus.

It is a beautiful example of invariance. Invariance in mathematics usually refers to a quantity that remains unchanged despite changes to other attributes related to that quantity. In this example, we could start drawing right angled triangles from the common centre, and use a little algebra to find the surface area — but there is an easier way. Those of you familiar with this problem will know that for a fixed chord length, the inner and outer radii can vary without changing the area of the annulus. This means that we can reduce the inner radius to zero, so that the chord becomes a diameter of the outer circle. Voila! The area of the annulus is the area of the circle with that diameter!

Invariance is an attractive notion, particularly when it strikes unexpectedly in my classroom. It is generally pursued with some vigour there and then. There is nothing quite like following a mathematical trail with a reasonably talented class in search of understanding.

Cardinal and parent circles

Just recently I had reason to refer to a book with the title "Fallacies in Mathematics" by E. A. Maxwell (1959). It is a wonderful collection of errors and pitfalls, compiled from classroom experiences. In chapter 11, one "miscellaneous howler" caught my attention. It states (albeit with a few of my own labelling changes in Figure 2): "To prove that if *BD* is a chord of a circle perpendicular to a diameter *AC* and meeting it in *Q*, then the sum of the areas of the circles on *AQ, BQ, CQ, DQ* as diameters, is equal to the area of the whole circle."
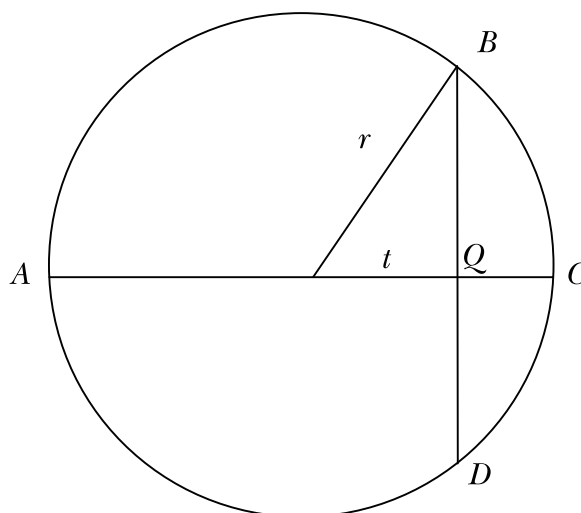

Figure 2

It then proceeds to give an erroneous "student proof" as follows:

The sum of the areas is
$$\frac{1}{4}\pi\left(AQ^2 + BQ^2 + CQ^2 + DQ^2\right)$$
$$= \frac{1}{4}\pi Q^2\left(A + B + C + D\right)$$

Since *A*, *B*, *C* and *D* are all points on the circumference; the sum of the areas is equal to the area of the whole circle.

To be perfectly honest, I did not know whether the theorem was true or not, but I thought it worthy of investigation. It was clearly referring to some sort of invariance property. I drew Figure 2 (a good starting point!) and called the four circles on the diameters *AQ, BQ, CQ, DQ* cardinal circles because of their orientation around the centre point *Q*.

From Figure 2, $\qquad\qquad\qquad BQ^2 = DQ^2 = r^2 - t^2$

So $\qquad\qquad\dfrac{1}{4}\pi\left(AQ^2 + BQ^2 + CQ^2 + DQ^2\right)$

$$= \frac{1}{4}\pi\left\{(r+t)^2 + \left(r^2 - t^2\right) + (r-t)^2 + \left(r^2 - t^2\right)\right\}$$

The right hand side simplifies to πr^2 which means that the theorem is true.

As a special case, (see Figure 3) if *AC* and *BD* were diameters, each *cardinal* circle would be one quarter of the *parent* circle (another name I invented!). The radius of the *parent* was the diameter of each *cardinal* circle.
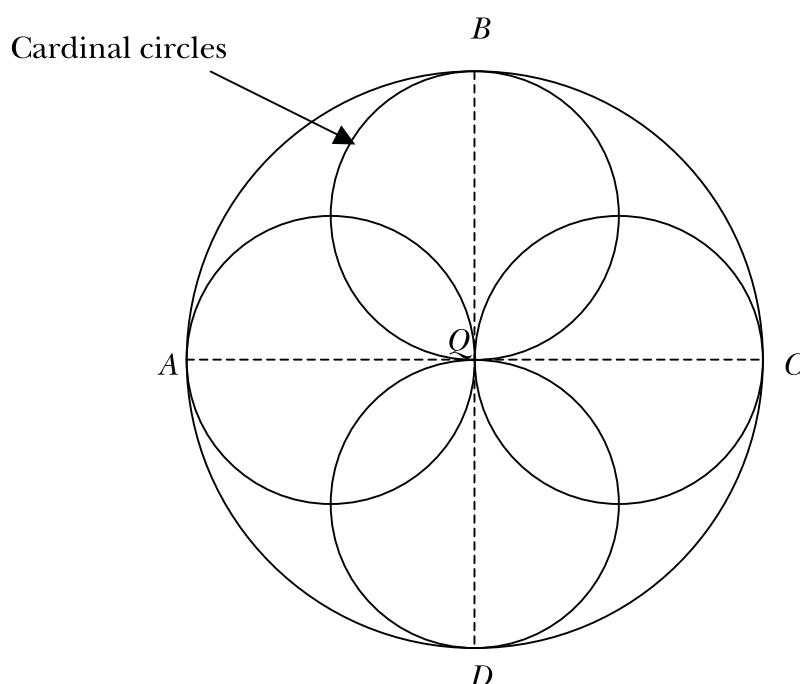

Figure 3

Generalising

What if *AC* and *BD* were *any* mutually perpendicular chords? How far could I take this invariance property?

Let us redraw the diagram in a more generalised way. Take any circle centre *O* (the parent circle) and any point *Q* within it. Draw two orthogonal chords passing through *Q* and cutting the parent circle at points *A*, *B*, *C* and *D* as shown in Figure 4.

We wish to prove that the sum of the areas of the circles with diameters *AQ*, *BQ*, *CQ* and *DQ* is equal to the area of the *parent* circle.
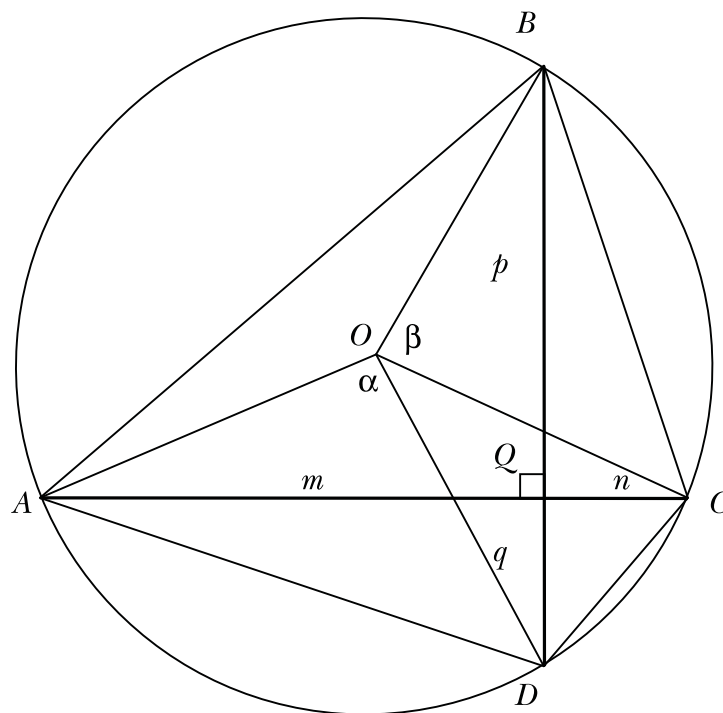
Figure 4

Then,

$m^2 + q^2 = AD^2 = 2r^2(1 - \cos\alpha)$ (Pythagoras' theorem and the cosine rule)

$n^2 + p^2 = BC^2 = 2r^2(1 - \cos\beta)$ (Pythagoras' theorem and the cosine rule)

Therefore $\qquad m^2 + p^2 + n^2 + q^2 = 2r^2(2 - \cos\alpha - \cos\beta)$

And finally $\qquad \dfrac{1}{4}\pi\left(m^2 + p^2 + n^2 + q^2\right) = \pi r^2\left\{1 - \dfrac{1}{2}\left(\cos\alpha + \cos\beta\right)\right\}$

This implies that if we can prove that $\cos\alpha + \cos\beta = 0$, the theorem is true for any Q as an internal point. Note that $\angle BAC = \frac{1}{2}\beta$ and $\angle ABD = \frac{1}{2}\alpha$ and since $\angle AQB = 90°$ it follows that $\frac{1}{2}\alpha + \frac{1}{2}\beta$ is 90° and that $\alpha + \beta = 180°$.

Therefore $\qquad\qquad \cos\alpha + \cos\beta = \cos(180 - \beta) + \cos\beta$

$$= -\cos\beta + \cos\beta$$

$$= 0.$$

If you are still with me, stop reading and go to www.canberramaths.com.au and click on: Resources> Post-Primary Classroom> Computers (ICTs) and then click on the link to the Excel program "Putting a spin on invariance in circles." Use the spinners to move Q, and watch the effect. Figure 5 shows the Excel spinner program.

The spinner program allows you to move Q either horizontally or vertically, and the cardinal circles enlarge, diminish, or even disappear accordingly. You can also play with the parent radius to see different effects. Figure 5 shows Q chosen 6 units to the right of centre, and 7 units down from centre of the parent circle whose radius is 16 units.
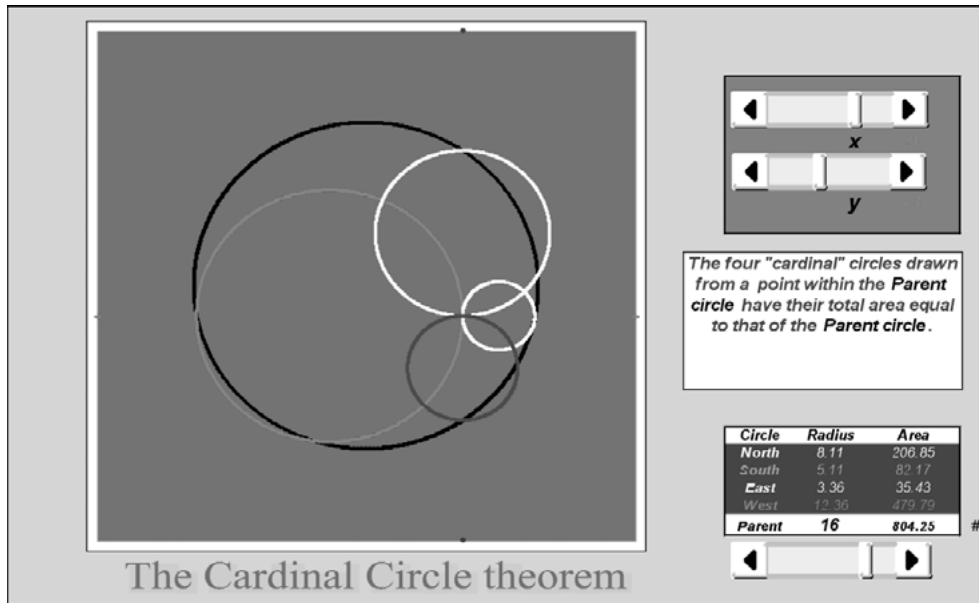
Figure 5. Screen shot of the spinner program allowing investigation of the cardinal circle theorem.

Extending the theorem

We also wish to investigate whether we can extend the theorem to include *Q* either on or outside the circle. When *Q* is on the circumference, two of the four cardinal circles cease to exist. When *Q* is outside the circle, the straight lines can either be secants, tangents, or completely miss the circle. Irrespective of whether *Q* is inside or outside the circle, if each line is either a secant or a tangent, will the sum of the areas of all four cardinal circles equal the area of the parent circle?

When *Q* lies on the circumference, *AB* becomes a diameter as shown in Figure 6.
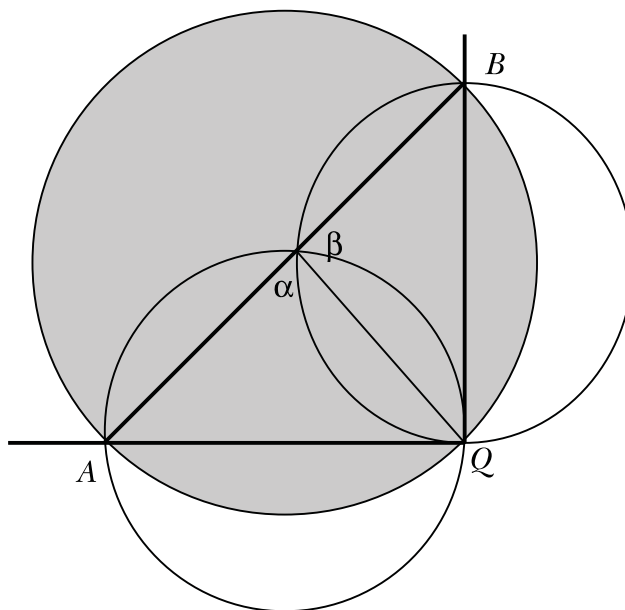

Figure 6. The case where Q is on the circumference.

We see immediately the circular equivalent to Pythagoras' theorem. Figure 6 shows the two remaining cardinal circles. Clearly $\alpha + \beta = 180°$, and the area sum is the area of the parent circle. This implies that the *area of semicircle on the hypotenuse is equal to the sum of the semicircles on the other two sides.* Since this is true, the combined area of the lunulae on *AQ* and *BQ* (the unshaded pieces) is equal to the area of the triangle *ABQ*. As an historical note, many mathematicians of ancient times (including Hippocrates of whom the lunulae are named after) continued a hopeless search for a method of squaring the circle, because of the remarkable fact that π does not occur in the lunulae formula.

To tackle the cases where *Q* is external, we need to first consider those instances where *Q* lies inside a circumscribed square as shown in Figure 7. This ensures that each perpendicular line is either a chord or a tangent. Figure 7 shows only one of four possible cases — where both lines become chords.
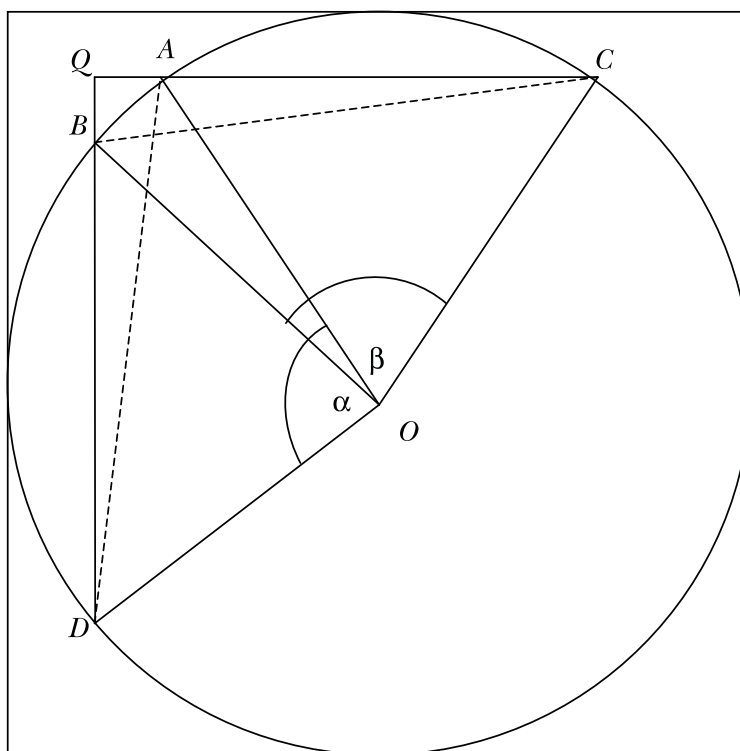

Figure 7. The case where Q is external to the circle but inside a circumscribed square.

Note also that the *A* and *B* labelling swapped around. This is because when *Q* moved from inside to on to outside, the points *A* and *B* crossed over. This becomes significant in the proof because now angles α and β overlap, as shown in Figure 7. Everything remains the same, except that we have to check to see whether $\alpha + \beta = 180°$.

Now reflex $\angle BOC = 360° - \beta$, which means $\angle BAC = 180° - \frac{1}{2}\beta$. This in turn means $\angle QAB$ is $\frac{1}{2}\beta$, and by similar argument $\angle QBA$ is $\frac{1}{2}\alpha$, implying *once again* that $\alpha + \beta = 180°$. That is to say that provided *QC* and *QD* are chords or tangents, the theorem remains true.

When *Q* is outside the circumscribed square, the theorem fails. There are

a number of cases that can be identified by the spinner program. Figure 8 shows one of these.
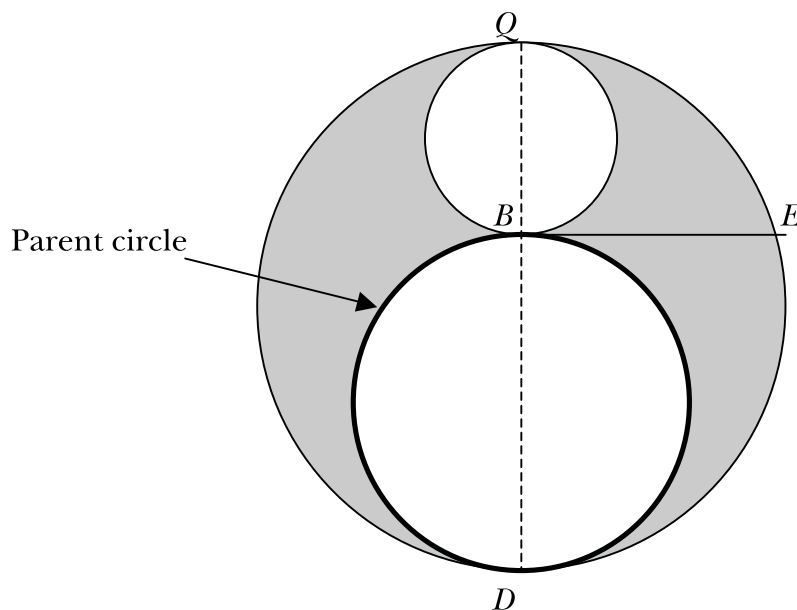

Figure 8. The case where Q is above the parent circle.

Here *Q* is above the parent circle, and the area of the two cardinal circles is clearly larger than the parent circle. In fact what we have created is two *arbolae* (literally "cobblers' knives;" Gellert, Kustner, Hellwich & Kastner, 1975) on either side of the line *QD*. These are the spaces between the large cardinal circle and the two circles it contains, shown shaded in Figure 7. The area of each arbolas is the area of a circle whose diameter is the horizontal line segment *BE* from the point of contact of the inner circles out to the large cardinal circle.

Invariance is one of those fascinating attractions of mathematics, perhaps not emphasised nearly enough in classrooms. Next time that you stumble across an invariance property, I urge you to pull it apart with your students.

References

Maxwell, E. A. (1959). *Fallacies in mathematics.* Cambridge, UK: Cambridge University Press.

Gellert, W., Kustner, H., Hellwich, M. & Kastner, H. (1975). *The VNR concise encyclopedia of mathematics.* New York: Van Nostrand Reinhold.